Budker Institute of Nuclear Physics

BINP 96-49
July 1996

Relativistic corrections to the positronium decay rate revisited

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Abstract

We rederive here in a simple and transparent way the master formula for the dominant part of large relativistic corrections to the positronium decay rate.

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1. The strong disagreement between the experimental value of the orthopositronium decay rate [1]

$$\Gamma^{o-Ps}_{exp} = 7.0482(16)\ \mu s^{-1} \tag{1}$$

and its theoretical value which includes the order α and $\alpha^2 \log(1/\alpha)$ corrections [2–5]

$$\Gamma^{o-Ps}_{th} = m\alpha^6\ \frac{2(\pi^2-9)}{9\pi}\ \left[1 - 10.28\frac{\alpha}{\pi} - \frac{1}{3}\alpha^2 \log\frac{1}{\alpha}\right]\ =\ 7.03830\ \mu s^{-1} \tag{2}$$

is a real challenge to the modern QED. For the disagreement to be resolved within the QED framework, the correction $\sim (\alpha/\pi)^2$, which has not been calculated completely up to now, should enter the theoretical result (2) with a numerical factor 250(40), which may look unreasonably large.

Such a hope is not as unreasonable however. One class of large second-order corrections arises as follows [6]. The large, -10.28, factor at the α/π correction to the decay rate (see (2)) means that the factor at the α/π correction to the decay amplitude is roughly 5. Correspondingly, this correction squared contributes about $25(\alpha/\pi)^2$ to the decay rate. Indeed, numerical calculations [7, 8] give factor 28.86 at $(\alpha/\pi)^2$ in this contribution.

One more class of potentially large contributions to the positronium decay rate is relativistic corrections. A simple argument in their favour is that the corresponding parameter $(v/c)^2 \sim \alpha^2$ is not suppressed, as distinct from that of usual second-order radiative corrections, $(\alpha/\pi)^2$, by the small factor $1/\pi^2 \sim 1/10$.

This problem was addressed in Refs. [9–11]. However, the discrepancy between the results obtained in [9], on one hand, and in [10, 11], on another, is huge. While according to [9], the relativistic correction constitutes (in "radiative" units, with $1/\pi$) $24.6(\alpha/\pi)^2$, the results of [10, 11], obtained in different techniques, are in a reasonable agreement between themselves: $46(3)\,(\alpha/\pi)^2$ [10] and $41.9\,(\alpha/\pi)^2$ [11]. The main source of this discrepancy can be traced back to the different treatment of $(v/c)^2$ arising in the expansion of the annihilation kernel. While the prescription of [9] is effectively (see their formulae (12), (18))

$$(v/c)^2\ \longrightarrow\ -\alpha^2/4, \tag{3}$$

our master formula is

$$(v/c)^2\ \longrightarrow\ -3\alpha^2/4. \tag{4}$$

Let us mention here that this our recipe refers to the relativistic correction to the annihilation kernel itself; as to the phase space correction, we use in it the prescription (3).

In view of the mentioned discrepancy, we believe that it is instructive to present an alternative derivation of formula (4), more obvious and transparent than our original one.

2. When calculating the decay amplitude, we have to integrate the annihilation kernel $M(\vec{p})$ over the distribution of the electron and positron three-momenta $\vec{p}$. We address in this note the relativistic corrections to $M(\vec{p})$ only, i.e., we take as the ground-state wave function $\psi(p)$ the nonrelativistic one. Then the decay amplitude is

$$\int \frac{d\vec{p}}{(2\pi)^3}\,\psi(\vec{p})\,M(\vec{p}) \;=\; \int \frac{d\vec{p}}{(2\pi)^3}\,\frac{8\sqrt{\pi a^3}}{(p^2a^2+1)^2}\,M(\vec{p}), \tag{5}$$

where $a = 2/m\alpha$ is the positronium Bohr radius. To lowest approximation in v/c the kernel $M(0)$ is independent of those momenta and we are left with

$$M(0)\int \frac{d\vec{p}}{(2\pi)^3}\,\frac{8\sqrt{\pi a^3}}{(p^2a^2+1)^2} \;=\; M(0)\,\psi(\vec{r}=0). \tag{6}$$

Thus, in the limit $p \to 0$ we obtain the common prescription: the positronium decay rate is proportional to $|\psi(r=0)|^2$.

However, already to first order in $(p/m)^2$ the momentum integral

$$\int d\vec{p}\,(p/m)^2\frac{8\sqrt{\pi a^3}}{(p^2a^2+1)^2} \tag{7}$$

linearly diverges at $p \to \infty$, which precludes the straightforward evaluation of these relativistic corrections.

The crucial observation is that the true relativistic expression for the annihilation kernel does not grow up at $p \to \infty$, as distinct from its expansion in p/m. So, the initial integral (5) in fact converges.

When treating relative corrections to the decay amplitude, it is convenient to single out from it the factor $\psi(r=0) = (\pi a^3)^{-1/2}$, and a trivial overall dimensional factor from $M(\vec{p})$. So, from now on we investigate, instead of (5), the following expression:

$$\int \frac{d\vec{p}}{(2\pi)^3}\,\frac{8\pi a^3}{(p^2a^2+1)^2}\,M(\vec{p}) \tag{8}$$

with dimensionless $M(\vec{p})$.

Let us consider first an auxiliary integral

$$\int \frac{d\vec{p}}{(2\pi)^3}\,\frac{8\pi a^3}{p^4a^4}\,[M(\vec{p})-M(0)], \tag{9}$$

which converges both at low and high p. After the angular integration, the dimensionless kernel $M(\vec{p})$ depends on the ratio $y^2=(p/m)^2$ only, and the expression (9) reduces to

$$\frac{4\alpha}{\pi}\int_0^\infty \frac{dy}{y^2}\,[M(y^2)-M(0)]. \tag{10}$$

In other words, this auxiliary integral is of first order in α/π, and therefore of no interest for our problem. This is a first-order radiative correction absorbed already by $-10.28\alpha/\pi$ in (2) (for orthopositronium). In fact, we have neglected in this argument the kernel dependence on the positronium binding energy (it certainly exists at least in the noncovariant perturbation theory we were starting from in [10]). But corrections effectively neglected in this way, are of higher odd powers in α.

So, expression (9) can be used as a regulator, and in the now rapidly converging integral

$$\int \frac{d\vec{p}}{(2\pi)^3}\,8\pi a^3\left[\frac{1}{(p^2a^2+1)^2}\;-\;\frac{1}{p^4a^4}\right][M(\vec{p})-M(0)] \tag{11}$$

we can safely expand $M(\vec{p})$ up to $(p/m)^2$ included. In this way we obtain

$$\int \frac{d\vec{p}}{(2\pi)^3}\,8\pi a^3\left[\frac{1}{(p^2a^2+1)^2}\;-\;\frac{1}{p^4a^4}\right]\left(\frac{p}{m}\right)^2\;=\;-\,\frac{3}{4}\,\alpha^2. \tag{12}$$

This is an alternative derivation of the master formula (4) used in our article [10]. In our opinion, this derivation leaves no doubts in the correctness of this prescription.

One of us (I.Kh.) is grateful to P. Labelle for discussions.

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